

Separation of Variables for a Lattice Integrable System and the Inverse Problem

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We investigate the relation between the local variables of a discrete integrable lattice system and the corresponding separation variables, derived from the associated spectral curve. In particular, we have shown how the inverse transformation from the separation variables to the discrete lattice variables may be factorized as a sequence of canonical transformations.

KEY WORDS: separation of variables; nonlinear integrable system; inverse scattering.

1. INTRODUCTION

Of late there has been a great deal of interest in discrete integrable systems, owing to their numerous interesting properties. One such property, which has emerged lately, is the existence of the so called spectral curve associated with an integrable system (Sklyanin, 1999). The latter in turn is closely connected with the motion of separability. Indeed, it is well known that the technique of separation of variables provides an invaluable tool for the analytic construction of action-angle variables of an integrable system (Sklyanin, 1999). Furthermore, it may be shown that the locus of points, given by the separation variables, define in fact the spectral curve.

On the other hand, the concept of an r -matrix algebra marks a major development in theory of nonlinear integrable systems especially in regard to their Hamiltonian nature (Faddeev and Takhtajan, 1988). It provides an elegant formalism for proving the Poisson involutiveness of the integral of motions for such

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systems. However, it was only in the eighties that the connection between the r -matrix formalism and separation variables of such systems clearly emerged as a result of the efforts of Komarov (1982), Sklyanin (1995) and other researchers of the St. Petersburg School.

In this communication, we examine the nature of the relationship between the separation variables of a discrete lattice integrable system and the corresponding local lattice variables. In general, the nature of the transformation relations between these two sets of variables is rather complicated and an explicit determination of the generating function proves to be rather daunting. The existence of the latter is essential to guarantee the canonical nature of the transformation. However, recently (Christiansen *et al.*, 1993) has shown that, not only does the inverse transformation between the separation variables and the local variables exist, but also that these transformations can even be broken down into a sequence of transformations each with a generating function.

The organization of the paper is as follows. In Section 2, we describe the basic features of the discrete model under consideration. The r -matrix algebra of the monodromy matrix is explicitly solved, as also the leading order structures of its elements. In Section 3, we introduce the spectral curve and separation representation for the elements of the monodromy matrix. In Section 4, which constitutes the main body of the paper, we derive explicitly the inverse separating map between the separation variables and the local lattice variables defining our system. The canonical nature of this mapping is displayed by working out the corresponding generating function.

2. FORMULATION

We begin by considering a discrete integrable system which is described by a local Lax operator $l_n(u)$ of the form

$$l_n(u) = \begin{pmatrix} u + q_n p_n & -p_n \\ q_n^2 p_n & u - q_n p_n \end{pmatrix} \quad (2.1)$$

Here u is the complex spectral parameter and q_n and p_n are the nonlinear variables defined at the n th lattice points. They satisfy the canonical Poisson bracket

$$\{q_k, q_j\} = \{p_k, p_j\} = 0 \quad \text{for all } k, j = 1, 2, \dots, n \text{ and } \{q_k, p_j\} = \delta_{kj} \quad (2.2)$$

It is a matter of simple computation to show that

$$\{l_n(u) \otimes l_m(v)\} = [r(u-v), l_n(u) \otimes l_m(v)] \delta_{mn} \quad (2.3)$$

where $r(u - v)$ represents the classical r -matrix given by

$$r(u) = \frac{P}{u} = \frac{1}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with P being the permutation matrix satisfying $P^2 = I$. We define the monodromy matrix $T_N(u)$ as the ordered product of $l_n(u)$'s over the entire lattice, i.e.

$$T_N(u) = \prod_{n=N}^1 l_N(u) = l_N(u)l_{N-1}(u) \cdots l_2(u)l_1(u) \tag{2.4}$$

Upon using (2.3) it is easy to deduce that

$$\{T_N(u), T_N(v)\} = [r(u - v), T_N(u) \otimes T_N(v)] \tag{2.5}$$

Consequently, we infer that structurally the Poisson algebra of $l_n(u)$ and $T_N(u)$ are identical. In the literature on nonlinear integrable systems Eq. (2.5) is usually referred to as the quadratic r -matrix algebra. In case of 2×2 local Lax operators, the monodromy matrix is itself a 2×2 matrix which may formally be written as,

$$T_N(u) = \prod_{n=N}^1 l_n(u) \equiv \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix} \tag{2.6}$$

The Sklyanin algebra (2.5) then takes the following explicit form, namely

$$\begin{aligned} \{A_N(u), A_N(v)\} &= \{B_N(u), B_N(v)\} = \{C_N(u), C_N(v)\} = \{D_N(u), D_N(v)\} = 0, \\ \{C_N(u), A_N(v)\} &= \frac{1}{u - v} \{A_N(u)C_N(v) - A_N(v)C_N(u)\}, \\ \{A_N(u), D_N(v)\} &= \frac{1}{u - v} \{C_N(u)B_N(v) - C_N(v)B_N(u)\}, \\ \{B_N(u), A_N(v)\} &= \frac{1}{u - v} \{B_N(u)A_N(v) - B_N(v)A_N(u)\}, \\ \{C_N(u), D_N(v)\} &= \frac{1}{u - v} \{C_N(u)D_N(v) - C_N(v)D_N(u)\}, \\ \{B_N(u), D_N(v)\} &= \frac{1}{u - v} \{D_N(u)B_N(v) - D_N(v)B_N(u)\}, \end{aligned} \tag{2.7}$$

From the definition of monodromy operator in (2.6) and the form of the local Lax operator as given by (2.1) by explicit calculation obtain the leading order terms of the matrix elements $A_N(u)$, $B_N(u)$ etc. These are as follows

$$A_N(u) = u^N + \left(\sum_{i=1}^N q_i p_i \right) u^{N-1} + \dots \tag{2.8a}$$

$$B_N(u) = \left(- \sum_{i=1}^N p_i \right) u^{N-1} + \dots \quad (2.8b)$$

$$C_N(u) = \left(\sum_{i=1}^N q_i^2 p_i \right) u^{N-1} + \dots \quad (2.8c)$$

$$D_N(u) = u^N - \left(\sum_{i=1}^N q_i p_i \right) u^{N-1} + \dots \quad (2.8d)$$

Note that $\det l_n(u) = u^2$ so that $\det T_N(u) = u^{2N}$. Now if at each lattice site an inhomogeneity parameter c_n is introduced, then the local Lax operator assumes the following form:

$$l_n(u - c_n) = \begin{pmatrix} u - c_n + q_n p_n & -p_n \\ q_n^2 p_n & u - c_n - q_n p_n \end{pmatrix}, \quad (2.9)$$

and consequently the monodromy matrix

$$T_N(u, \{c_n\}) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}$$

now involves the inhomogeneity parameters $\{c_n\}_{n=1}^N$. Its elements then are as given below

$$A_N(u, \{c_n\}) = u^N + \left(\sum_{i=1}^N q_i p_i - \sum_{i=1}^N c_i \right) u^{N-1} + \dots \quad (2.10a)$$

$$B_N(u, \{c_n\}) = \left(- \sum_{i=1}^N p_i \right) u^{N-1} + \dots \quad (2.10b)$$

$$C_N(u, \{c_n\}) = \left(\sum_{i=1}^N q_i^2 p_i \right) u^{N-1} + \dots \quad (2.10c)$$

$$D_N(u, \{c_n\}) = u^N - \left(\sum_{i=1}^N q_i p_i + \sum_{i=1}^N c_i \right) u^{N-1} + \dots \quad (2.10d)$$

Thus, in general A_N, D_N are polynomials of degree N , while B_N and C_N are polynomials of degree $N - 1$ in the spectral parameter. Furthermore, in this case

$$\det l_n(u - c_n) = (u - c_n)^2$$

so that

$$\det T_N(u, \{c_n\}) = \prod_{n=1}^N (u - c_n)^2 \quad (2.11)$$

The involutive, independent integrals of the system are now obtained as the coefficients of trace of the monodromy matrix. That is,

$$\begin{aligned} \text{tr} T_N(u, \{c_n\}) &= A_N(u, \{c_n\}) + D_N(u, \{c_n\}) \\ &= 2u^N + H_1 u^{N-1} + H_2 u^{N-2} \\ &\quad + \dots + H_{N-1} u + H_N \end{aligned} \quad (2.12)$$

For a three particle system i.e., $N = 3$

$$\begin{aligned}
 H_1 &= -2(c_1 + c_2 + c_3) \\
 H_2 &= 4(c_3c_1 + c_1c_2 + c_2c_3) + 2(k_3k_2 + k_2k_1 + k_1k_3) - k_3\rho_3p_2 \\
 &\quad - k_3\rho_3p_1 - k_2\rho_2p_3 - k_2\rho_2p_1 - k_1\rho_1p_3 - k_1\rho_1p_2
 \end{aligned}$$

Note that throughout this paper, we shall assume periodic boundary conditions i.e., $q_{n+N} = q_n$ and $p_{n+N} = p_n$

3. SPECTRAL CURVE AND SEPARATION REPRESENTATION

In case of Liouville integrable systems, the technique of separation of variables play a vital role in construction of analytic expressions for the action angle variables. However, it was only in the early eighties that the connection between separation variables and the r -matrix formalism of the quantum inverse scattering method became evident. The fact that for 2×2 Lax matrices, the separation variables are given by the zeros of the off diagonal of $T_N(u)$ was first observed by Komarov (1982). The technique was investigated in great detail by Sklyanin (1995) and was applied to several models, such as the Goryachev–Chaplygin top, the Neuman system (Sklyanin, 1984) etc. Indeed, its subsequent application to the $O(4)$ Kowalevski top, by Kuznetsov, represents the culmination in a sense, of this remarkable technique.

We shall briefly outline the connection between separation variables and the elements of the so-called spectral curve, which is explained below. It is well known that, an integrable Hamiltonian system possessing N independent integrals of motion in involution, is separable, if there exists a set of Darboux coordinates say $(u_i, v_i)_{i=1}^N$ which satisfy a set of equations of the form

$$f(u_i, v_i, I_1, I_2, \dots, I_N) = 0 \quad (i = 1, 2, \dots, N) \tag{3.1}$$

These equations arise from the equation of an associated spectral curve Γ defined by the following equation

$$\det(zI - T_N(u)) = z^2 - P(u)z + Q(u) = 0, \tag{3.2}$$

on the (u, z) plane. Here, $P(u)$ and $Q(u)$ are given by $\text{tr}(T_N(u))$ and $\det(T_N(u))$, respectively. In our case, we note that $B_N(u)$ i.e., the upper off diagonal element of $T_N(u)$ is a polynomial of degree $(N - 1)$, so that it can have at most $(N - 1)$ distinct zeros, which we shall denote by $u_i (i = 1, 2, \dots, N - 1)$. One may show that there exists a system of differential equations for these zeros. It is easy to see that at the zeros of $B_N(u)$, viz., $u_i (i = 1, 2, \dots, N - 1)$ the monodromy matrix

becomes triangular, i.e.,

$$T_N(u_i) = \begin{pmatrix} A_N(u_i) & 0 \\ C_N(u_i) & D_N(u_i) \end{pmatrix} \tag{3.3}$$

and therefore its eigenvalues are the diagonal elements $A_N(u_i)$ and $D_N(u_i)$, respectively. If $z_i \equiv A_N(u_i), i = 1, 2, \dots, N - 1$, then it is obvious that the set of points $(u_i, z_i)_{i=1}^N$ of the $(u - z)$ plane must lie on the spectral curve Γ_i

$$z_i^2 - P(u_i)z_i + Q(u_i) = 0, \quad i = 1, 2, \dots, N - 1. \tag{3.4}$$

For the remaining part of this communication, we shall set $z_i = e^{-v_i}$ and shall show that the pair (u_i, v_i) constitute a set of Darboux co-ordinates satisfying the canonical brackets

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \quad \{u_i, v_j\} = \delta_{ij} \quad i, j = 1, 2, \dots, N - 1. \tag{3.5}$$

Let us further define two quantities as follows:

$$u_N \equiv \frac{\sum_{i=1}^N q_i p_i - c_i}{-\sum_{i=1}^N p_i}, \quad v_N \equiv -\sum_{i=1}^N p_i,$$

satisfying $\{u_N, v_N\} = 1$ Thus altogether we have a total of N pairs of Darboux coordinates $(u_i, v_i)_{i=1}^N$. Having defined the above separation variables, we next proceed to the construction of a representation of the quadratic r -matrix algebra (2.5) in terms of these separation variables. This is easily accomplished by using Lagrange interpolation. For instance, we may express $B_N(u)$ in terms of its zeros u_1, u_2, \dots, u_{N-1} as

$$B_N(u) = u_N \prod_{n=1}^{N-1} (u - u_n). \tag{3.6}$$

Furthermore, assuming these zeros to be mutually distinct we may express any polynomial $F(u)$ of degree less than $(N - 1)$ in the following manner

$$\frac{F(u)}{B_N(u)} = \sum_{i=1}^{(N-1)} \frac{F(u_i)}{B'_N(u_i)(u - u_i)}, \tag{3.7}$$

while if $F(u)$ has degree $\geq (N - 1)$, then

$$\frac{F(u)}{B_N(u)} = \sum_{i=1}^{(N-1)} \frac{F(u_i)}{B'_N(u_i)(u - u_i)} + G(u) \tag{3.8}$$

where $G(u)$ is a polynomial of appropriate degree. In the present case since $A_N(u)$ and $D_N(u)$ are polynomials of degree N , we have the following representations

for them,

$$\begin{aligned}
 &A_N(u) \\
 &= B_N(u) \left[\frac{u + \left(u_N v_N + \sum_{i=1}^{(N-1)} u_i\right)}{u_N} + \sum_{i=1}^{N-1} \frac{e^{-v_i}}{B'_N(u_i)(u - u_i)} \right] \tag{3.9}
 \end{aligned}$$

$$\begin{aligned}
 &D_N(u) \\
 &= B_N(u) \left[\frac{u + \left(u_N v_N + \sum_{i=1}^{(N-1)} u_i + 2 \sum_{i=1}^N c_i\right)}{u_N} + \sum_{i=1}^{N-1} \frac{\det T_N(u_i) e^{v_i}}{B'_N(u_i)(u - u_i)} \right] \tag{3.10}
 \end{aligned}$$

since

$$\det T_N(u_i) = A_N(u_i) D_N(u_i) = e^{-v_i} D_N(u_i) \tag{3.11}$$

The representation for $C_N(u)$ follows from the relation

$$\det T_N(u) = A_n(u) D_N(u) - B_N(u) C_N(u)$$

so that

$$C_N(u) = \frac{A_N(u) D_N(u) - \det T_N(u)}{B_N(u)} \tag{3.12}$$

with $A_N(u)$, $D_N(u)$, and $B_N(u)$ as given by the relations (3.9), (3.10), and (3.6) while

$$\det T_N = \prod_{n=N}^1 \det l_n(u - c_n) = \prod_{n=1}^N (u - c_n)^2.$$

We shall now proceed to show that the above representations for the elements of $T_N(u)$ does indeed reproduce the quadratic r -matrix algebra. For example,

$$\begin{aligned}
 \{B_N(u), A_N(v)\} &= \left\{ B_N(u), B_N(v) \left[\frac{v + (u_N v_N + \sum_{i=1}^{(N-1)} u_i)}{v_N} \right. \right. \\
 &+ \left. \left. \sum_{i=1}^{N-1} \frac{e^{-v_i}}{B'_N(u_i)(v - u_i)} \right] \right\} = B_N(v) \left\{ B_N(u), \frac{v + (u_N v_N + \sum_{i=1}^{(N-1)} u_i)}{v_N} \right. \\
 &+ \left. \sum_{i=1}^{N-1} \frac{e^{-v_i}}{B'_N(u_i)(v - u_i)} \right\}
 \end{aligned}$$

since $\{B_N(u), B_N(v)\} = 0$.

$$\begin{aligned}
 &= B_N(v)\{v_N, u_N\} \prod_{n=1}^{N-1} (u - u_i) + B_N(v) \sum_{i=1}^N \\
 &\quad - 1v_N \prod_{k=1}^{i=1} (u - u_k)\{-u_i, e^{v_i}\} \frac{1}{B'_N(u_i)(v - u_i)} \prod_{k=i+1}^{N-1} (u - u_k) \\
 &= -\frac{B_N(v)B_N(u)}{v_N} + \frac{B_N(v)B_N(u)}{u - v} \sum_{i=1}^{N-1} \frac{e^{-v_i}}{B'_N(u_i)} \left(\frac{1}{v - u_i} - \frac{1}{u - u_i} \right) \\
 &= -\frac{B_N(v)B_N(v)}{v_N} + \frac{B_N(u)B_N(v)}{u - v} \left(\frac{A_N(v)}{B_N(v)} - \frac{v + u_N v_N + \sum_{i=1}^{(N-1)} u_i}{u_N} \right. \\
 &\quad \left. - \frac{A_N(u)}{B_N(u)} + \frac{u + u_N v_N + \sum_{i=1}^{(N-1)} u_i}{v_N} \right) \\
 &\quad \times \{B_N(u), A_N(v)\} = \frac{1}{u - v} (B_N(u)A_N(v) - B_N(v)A_N(u)).
 \end{aligned}$$

It is also straight forward to show that $\det T_N(u)$ is a Casimir, i.e., its Poisson brackets with the elements of $T_N(u)$ vanish.

4. FACTORIZED SEPARATION

From the above discussion, it is obvious that we have at our disposal two sets of local variables – the first set consisting of $(q_i, p_i)_{i=1}^N$, in terms of which the local Lax matrices are represented and a second set consisting of the separation variables $(u_i, v_i)_{i=1}^N$. The two sets of variables being connected by the separation map S_i given by

$$B_N(u_i) = 0, \quad e^{v_i} = A_N(u_i) \quad i = 1, 2, \dots, (N - 1) \tag{4.1a}$$

while

$$u_N = \frac{\sum_{i=1}^N (q_i p_i - c_i)}{-\sum_{i=1}^N p_i}, \quad v_N = -\sum_{i=1}^N p_i. \tag{4.1b}$$

The concept of factorized separation has recently been introduced by Christiansen *et al.*, (1993) to obtain an inverse of the separation map. Indeed, the mapping S_N between the original local variables (p_i, q_i) and the separation variables is in effect a rather complicated canonical transformation and to obtain its generating function is a tedious task. The same also holds for the inverse mapping which relates the separation variables to the local variables appearing in the Lax operators. Christiansen *et al.*, (1993) made the remarkable observation that not only does the

inverse mapping S_N^{-1} exist but that it could also be factorized as a chain of more simpler, at least in principle, canonical transforms, thus leading to the solution of the inverse problem.

We note that the N -site monodromy matrix may also be expressed as

$$T_N(u) = l_N(u - c_N)T_{N-1}(u) = \begin{pmatrix} u - c_N + q_N p_N & -p_N \\ q_N^2 p_N & u - c_N - q_N p_N \end{pmatrix} T_{N-1}(u), \tag{4.2}$$

while

$$T_{N-1}(u) = l_N^{-1}(u - c_N)T_N(u) = \frac{\text{Adj}l_N(u - n)}{\det l_N(u - c_N)} T_N(u). \tag{4.3}$$

If we formally denote

$$T_{N-1}(u) = \begin{pmatrix} A_{N-1}(u) & B_{N-1}(u) \\ C_{N-1}(u) & D_{N-1}(u) \end{pmatrix}, \tag{4.4}$$

then it is obvious that $B_{N-1}(u)$ is a polynomial of degree $(N - 2)$ while $A_{N-1}(u)$ and $D_{N-1}(u)$ are polynomials of degree $(N - 1)$ in u . Moreover, by analogy with their earlier counterparts, let $\tilde{u}_j (j = 1, 2, \dots, (N - 2))$ denote the zeros of B_{N-1} , so that

$$B_{N-1}(u)(\tilde{u}_j) = 0, \quad j = 1, 2, \dots, (N - 2) \tag{4.5}$$

and let

$$e^{-v_j} = A_{N-1}(\tilde{u}_j), \quad j = 1, 2, \dots, (N - 2) \tag{4.6}$$

while

$$\tilde{u}_{N-1} = \frac{\sum_{i=1}^{N-1} (q_i p_i - c_i)}{-\sum_{i=1}^{N-1} p_i}, \quad \tilde{u}_{N-1} = -\sum_{i=1}^{N-1} p_i \tag{4.7}$$

As before $A_{N-1}(u)$ and $D_{N-1}(u)$ may in view of (4.5) and (4.6) be expressed by Lagrange interpolation formulae in the undermentioned forms:

$$A_{N-1}(u) = B_{N-1}(u) \left[\frac{u + (\tilde{u}_N \tilde{v}_N + \sum_{j=1}^{(n-2)} \tilde{u}_j)}{\tilde{u}_N} + \sum_{j=1}^{N-2} \frac{e^{-\tilde{v}_j}}{B'_N(\tilde{u}_j)(u - \tilde{u}_j)} \right] \tag{4.8a}$$

$$D_{N-1}(u) = B_{N-1}(u) \left[\frac{u + (\tilde{u}_N \tilde{v}_N + \sum_{j=1}^{(n-2)} \tilde{u}_j + 2 \sum_{i=1}^N c_i)}{\tilde{u}_N} + \sum_{j=1}^{N-2} \frac{\det T_{N-1}(\tilde{u}_j) e^{\tilde{u}_j}}{B'_{N-1}(\tilde{u}_j)(u - \tilde{u}_j)} \right] \tag{4.8b}$$

while

$$B_{N-1}(u) = \tilde{u}_{N-1} \prod_{j=1}^{N-1} (u - \tilde{u}_j). \tag{4.9}$$

Let X_N denote the mapping from the separation variables $u_i, v_i (i = 1, 2, \dots, (N - 1))$ to the new separation variables $\tilde{u}_j, \tilde{v}_j (j = 1, 2, \dots, (N - 2))$ which parameterize $T_{N-1}(u)$ and into the pair of local variables q_N and p_N . From (4.3), since $\det l_N(u - c_N) = (u - c_N)^2$, we see that $T_{N-1}(u) = \frac{\text{Adj} l_N(u - c_N)}{(u - c_N)^2} T_N(u)$ has a second order pole at $u = c_N$. As $T_{N-1}(u)$ does not have any singularities, consequently upon equating the residue at $u = c_N$ to zero, we have

$$\lim_{u \rightarrow c_N} \frac{d}{du} [\text{Adj} l_N(u - c_N) T_N(u)] = 0, \tag{4.10}$$

which leads to the following relations:

$$\begin{pmatrix} -q_N p_N & p_N \\ -q_N^2 p_N & q_N p_N \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} A_N(c_N) & B_N(c_N) \\ C_N(c_N) & D_N(c_N) \end{pmatrix} \begin{pmatrix} \frac{d}{du} D_N(u) & -\frac{d}{du} B_N(u) \\ -\frac{d}{du} C_N(u) & \frac{d}{du} A_N(u) \end{pmatrix} \Big|_{u=c_N} \tag{4.11}$$

where

$$\Delta = \left[\frac{d}{du} A_N(u) \frac{d}{du} D_N(u) - \frac{d}{du} B_N(u) \frac{d}{du} C_N(u) \right] \Big|_{u=c_N}. \tag{4.12}$$

Hence

$$p_N = \frac{1}{\Delta} \left[-A_N(c_N) \frac{d}{du} B_N(u) + B_N(c_N) \frac{d}{du} A_N(u) \right] \Big|_{u=c_N} \tag{4.13}$$

and

$$q_N = \left[\frac{-C_N(u) \frac{d}{du} B_N(u) + D_N(u) \frac{d}{du} A_N(u)}{-A_N(u) \frac{d}{du} B_N(u) + B_N(u) \frac{d}{du} A_N(u)} \right] \Big|_{u=c_N} \tag{4.14}$$

Equations (4.13–4.14) determine the N th local variables in terms of the separation variables $(u_i, v_i)|_{i=1}^N$. To determine the remaining local variables we adopt a recursive procedure. To this end we shall first consider the mapping

$$X_N : (\vec{u}, \vec{v}) \mapsto (\vec{\tilde{u}}, \vec{\tilde{v}} | q_N, p_N), \tag{4.15}$$

where $\vec{u} = (u_1, u_2, \dots, u_N)$ etc. Next if we consider the mapping X_{N-1} given by,

$$X_{N-1} : (\vec{\tilde{u}}, \vec{\tilde{v}} | q_N, p_N) \mapsto (\vec{\tilde{\tilde{u}}}, \vec{\tilde{\tilde{v}}} | q_N, p_N; q_{N-1}, p_{N-1}) \tag{4.16}$$

then it is obvious that the composition of these two mappings,

$$x_{N-1} \circ X_N : (\vec{u}, \vec{v}) \mapsto (\vec{\tilde{\tilde{u}}}, \vec{\tilde{\tilde{v}}} | q_N, p_N; q_{N-1}, p_{N-1}) \tag{4.17}$$

Therefore the composition of N such mappings viz.,

$$X_1 \circ X_2 \circ \dots \circ X_{N-1} \circ X_N : (\vec{u}, \vec{v}) \mapsto (\vec{q}, \vec{p}) \tag{4.18}$$

leads to a transformation from the separation variables $(u_i, v_i)_{i=1}^N$ to the local variables $(q_i, p_i)_{i=1}^N$, which is precisely the definition of S_N^{-1} i.e.,

$$S_N^{-1} = X_1 \circ X_2 \circ \dots \circ X_{N-1} \circ X_N \tag{4.19}$$

Equation (4.19) represents the separation factorization chain of the inverse mapping from the separation variables to the local variables occurring in the Lax operators.

In the present case, the mapping X_N will be explicitly derived in the following manner (Christiansen *et al.*, 1993). From Eq. (4.2) we obtain

$$B_N(u) = (u - c_N + q_N p_N) B_{N-1}(u) - p_N D_{N-1}(u) \tag{4.20}$$

If this equation is evaluated at $u = \tilde{u}_j (j = 1, 2, \dots, N - 2)$ i.e., at the zeros of $B_{N-1}(u)$ then one obtains the following relation

$$B_N(\tilde{u}_j) = -p_N D_{N-1}(\tilde{u}_j) \quad j = 1, 2, \dots, N - 2 \tag{4.21}$$

$$B_N(\tilde{u}_j) = -p_N \{ \det T_{N-1}(\tilde{u}_j) e^{\tilde{v}_j} \} \quad j = 1, 2, \dots, N - 2 \tag{4.22}$$

$$= -p_N \prod_{k=1}^{N-1} (\tilde{u}_j - c_k)^2 e^{\tilde{v}_j}$$

Again from (4.3) we obtain

$$B_{N-1}(u) = \frac{1}{\det L_N(u - c_N)} [(u - c_N - q_N p_N) B_N(u) + p_N D_N(u)] \tag{4.23}$$

Its evaluation at $u = u_i$ gives

$$(u_i - c_N)^2 B_{N-1}(u_i) = p_N D_N(u_i) \quad i = 1, 2, \dots, N - 1 \tag{4.24}$$

Substituting expressions for $B_N(u)$ and $B_{N-1}(u)$ from (3), and (4.9), respectively, and $D_{N-1}(u)$ from (4.8b) into (4.20) and equating the leading order terms, we obtain from the coefficient of u^{N-1} ,

$$v_N = \tilde{v}_{N-1} - p_N. \tag{4.25}$$

However, it is also follows from the definitions of (u_N, v_N) and $(\tilde{u}_{N-1}, \tilde{v}_{N-1})$ as given in (4.1b) and (4.7), respectively that

$$u_N v_N = \sum_{i=1}^N (q_i p_i - c_i) \quad \text{and} \quad \tilde{u}_{N-1} \tilde{v}_{N-1} = \sum_{i=1}^{N-1} (q_i p_i - c_i), \tag{4.26}$$

so that their difference i.e.,

$$u_N v_N - \tilde{u}_{N-1} \tilde{v}_{N-1} = q_N p_N - c_N. \tag{4.27}$$

Now, from (4.25) and (4.27), we may solve for v_N and \tilde{v}_N to obtain,

$$v_N = \frac{p_N \tilde{u}_{N-1} + q_N p_N - c_N}{u_N - \tilde{u}_{N-1}} \tag{4.28a}$$

$$\tilde{v}_{N-1} = \frac{p_N u_N + q_N p_N - c_N}{u_N - \tilde{u}_{N-1}} \tag{4.28b}$$

Equating the coefficients of u^{N-2} in (4.20), we obtain

$$-v_N \sum_{i=1}^{N-1} = \tilde{v}_{N-1} \left\{ -\sum_{j=1}^{N-2} \tilde{u}_j + (q_N p_N - c_N) \right\} + p_N (\tilde{u}_N \tilde{v}_N + 2 \sum_{i=1}^{N-1} c_i)$$

leading to

$$p_N = \frac{-v_N \sum_{i=1}^{N-1} u_i + \tilde{v}_{N-1} (\sum_{j=1}^{N-2} \tilde{u}_j + c_N)}{(\tilde{u}_N \tilde{v}_N + 2 \sum_{i=1}^{N-1} c_i + q_N \tilde{v}_{N-1})} \tag{4.29}$$

Substituting the expressions for v_N and \tilde{v}_{N-1} from (4.28a,b) into (4.29) yields the following quadratic expressions for $p_N(q_N, \tilde{u}, \tilde{u}, \{c_i\})$ namely,

$$\begin{aligned} & p_N^2 (q_N + u_N)(q_N + \tilde{u}_{N-1}) + p_N [(q_N + \tilde{u}_{N-1}) (\sum_{i=1}^{N-1} u_i - c_N) \\ & - (q_N + u_N) (\sum_{j=1}^{N-2} \tilde{u}_j + c_N) + 2 \sum_{i=1}^{N-1} c_i (u_N - \tilde{u}_{N-1})] \\ & + c_N [c_N + \sum_{j=1}^{N-2} \tilde{u}_j - \sum_{i=1}^{N-1} u_i] = 0. \end{aligned} \tag{4.30}$$

Incidentally, if we set $c_i = 0$ for all $i = 1, 2, \dots, N$, then (4.30) leads to the following linear expressions for p_N

$$p_N = \left[\frac{\sum_{j=1}^{N-2} \tilde{u}_j}{q_N + \tilde{u}_{N-1}} - \frac{\sum_{i=1}^{N-1} u_i}{q_N + u_N} \right] \tag{4.31}$$

On the other hand, from (4.22), we get

$$e^{-\tilde{v}_j} = -\frac{p_N \prod_{k=1}^{N-1} (\tilde{u}_j - c_k)^2}{B_N(\tilde{u}_j)} = -\frac{p_N \prod_{k=1}^{N-1} (\tilde{u}_j - c_k)^2}{v_N \prod_{k=1}^{N-1} (\tilde{u}_j - v_k)}, \quad j = 1, 2, \dots, N - 2 \tag{4.32}$$

Upon substituting v_N from (4.28a), we get

$$e^{-\tilde{v}_j} = -\frac{p_N (u_N - \tilde{u}_{N-1})}{p_N (q_N + \tilde{u}_{N-1}) - c_N} \prod_{k=1}^{N-1} \frac{(\tilde{u}_j - c_k)^2}{(\tilde{u}_j - u_k)}, \quad j = 1, 2, \dots, N - 2 \tag{4.33}$$

Finally, from (4.24), we have upon using the expressions for $B_{N-1}(u_i)$ and $D_N(u_i)$ obtained from (4.9) and (3.11), respectively, the following relation,

$$e^{-\tilde{v}_j} = \frac{p_N}{\tilde{v}_{N-1}(u_i - c_N)^2} \frac{\prod_{m=1}^{N-1} (\tilde{u}_i - c_m)^2}{\prod_{j=1}^{N-2} (\tilde{u}_i - \tilde{u}_j)}, \quad i = 1, 2, \dots, N - 1. \quad (4.34)$$

when using (4.28b), to eliminate \tilde{v}_{N-1} , we get

$$e^{-\tilde{v}_j} = \frac{p_N(u_N - \tilde{u}_{N-1})}{p_N(q_N + \tilde{u}_{N-1}) - c_N} \frac{1}{(u_i - c_N)^2} \times \frac{\prod_{m=1}^{N-1} (\tilde{u}_i - c_m)^2}{\prod_{j=1}^{N-2} (\tilde{u}_i - \tilde{u}_j)}, \quad i = 1, 2, \dots, N - 1. \quad (4.35)$$

Thus, the mapping X_N is formally defined by the following relations

$$p_N^2(q_N + u_N)(q_N + \tilde{u}_{N-1}) + p_N \left[(q_N + \tilde{u}_{N-1}) \left(\sum_{i=1}^{N-1} u_i - c_N \right) - (q_N + u_N) \left(\sum_{j=1}^{N-2} \tilde{u}_j + c_N \right) + 2 \sum_{i=1}^{N-1} c_i(u_N - \tilde{u}_{N-1}) \right] + c_N \left[c_N + \sum_{j=1}^{N-2} \tilde{u}_j - \sum_{i=1}^{N-1} u_i \right] = 0 \quad (4.36)$$

$$e^{-v_i} = \frac{p_N(u_N - \tilde{u}_{N-1})}{p_N(q_N + u_N) - c_N} \frac{1}{(u_i - c_N)^2} \times \frac{\prod_{m=1}^{N-1} (\tilde{u}_i - c_m)^2}{\prod_{j=1}^{N-1} (\tilde{u}_i - \tilde{u}_j)}, \quad i = 1, 2, \dots, N - 1 \quad (4.37)$$

and

$$e^{-\tilde{v}_j} = -\frac{p_N(u_N - \tilde{u}_{N-1})}{p_N(q_N + \tilde{u}_{N-1}) - c_N} \prod_{k=1}^{N-1} \frac{(\tilde{u}_i - c_k)^2}{(\tilde{u}_j - \tilde{u}_k)}, \quad j = 1, 2, \dots, N - 2 \quad (4.38)$$

together with the relation

$$v_N = \frac{p_N(\tilde{u}_{N-1} + q_N) - c_N}{u_N - \tilde{u}_{N-1}}, \quad \tilde{v}_{N-1} = \frac{p_N(u_N + q_N) - c_N}{u_N - \tilde{u}_{N-1}} \quad (4.39)$$

For the special case when $c_i = 0$, for all $i = 1, 2, \dots, N$ these expressions reduce to the following

$$p_N = \left[\frac{\sum_{j=1}^{N-2} \tilde{u}_j}{q_N + \tilde{u}_{N-1}} - \frac{\sum_{i=1}^{N-1} u_i}{q_N + N} \right], \quad v_N = \frac{p_N(\tilde{u}_{N-1} + q_N)}{u_N - \tilde{u}_{N-1}}, \quad \tilde{v}_{N-1} = \frac{p_N(u_N + q_N)}{u_N - \tilde{u}_{N-1}} \quad (4.40a,b,c)$$

$$e^{-v_i} = \frac{u_N - \tilde{u}_{N-1}}{(q_N + u_N)} \frac{u_i^2}{\prod_{j=1}^{N-2} (u_i - \tilde{u}_j)} \quad i = 1, 2, \dots, N-1. \quad (4.40d)$$

$$e^{-v_j} = -\frac{u_N - \tilde{u}_{N-1}}{q_N + \tilde{u}_{N-1}} \prod_{k=1}^{N-1} \frac{(\tilde{u}_j^2)}{(\tilde{u}_j - u_k)} \quad j = 1, 2, \dots, N-2. \quad (4.40e)$$

It is easy to verify that the expressions given in Eq. (4.40 a–e) are derivable from the following generating function $F(q_N; \vec{u}, \vec{\tilde{u}})$, by means of the following mapping X_N :

$$\begin{aligned} v_i &= \frac{\partial F}{\partial u_i} \quad i = 1, 2, \dots, N. \\ \tilde{v}_j &= \frac{\partial F}{\partial \tilde{u}_j} \quad j = 1, 2, \dots, N-1. \\ p_N &= -\frac{\partial F}{\partial q_N}. \end{aligned} \quad (4.41)$$

with

$$\begin{aligned} F(q_N; \vec{u}, \vec{\tilde{u}}) &= \left[-\sum_{j=1}^{N-2} \tilde{u}_j \right] \log(q_N + \tilde{u}_{N-1}) + \left[\sum_{i=1}^{N-1} u_i \right] \log(q_N + u_N) \\ &+ \left[\sum_{j=1}^{N-2} \tilde{u}_j - \sum_{i=1}^{N-1} u_i \right] \log(u_N - \tilde{u}_{N-1}) \\ &+ 2(N-1) \sum_{j=1}^{N-2} \Theta(\tilde{u}_j) - 2(N-2) \sum_{i=1}^{N-1} \Theta(\tilde{u}_j) \\ &+ \sum_{i=1}^{N-1} \sum_{j=1}^{N-2} \Theta(u_i - \tilde{u}_j) + i\pi(N-2) \sum_{j=1}^{N-2} \tilde{u}_j \end{aligned} \quad (4.42)$$

where $\Theta(u) = \int \log(u) du + c(\text{constant})$. The generating function for the subsequent mapping X_{N-1} is obtained by replacing N by $N-1$ and so on. The above formulae thus describe explicitly the inverse separation map S_N^{-1} and by virtue of the existence of a generating function for each stage we conclude that the inverse separation map is canonical.

5. DISCUSSION

In this communication we have studied the nature of the inverse separation transformation between the local nonlinear variables of a discrete integrable system, and the corresponding separation variables of the system. The novel feature of the methodology used is the fact that this separation transformation can be factorized into a sequence of transformations, each with its corresponding generating

function. The latter clearly is indicative of the canonical nature of the separation transformation.

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REFERENCES

- Christiansen, P. L., Jorgensen, M. F., and Kuznetsov, V. B. (1993). *Letter of Mathematical Physics* **29**, 165–173.
- Faddeev, L., and Takhtajan, L. (1988). *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin.
- Komarov, I. V. (1982). *Theory of Mathematical Physics*, **50**(3), 4502–4509. Komarov, I. V. (1982). *Journal of Physics A* **15**(6), 1765–1773.
- Kuznetsov, V. B. (2001). *Kowalevski top revisited*, SI/0110012.
- Sklyanin, E. K. (1999). Sol-int/9903016.
- Sklyanin, E. K. (1999). Sol-int/990301.
- Sklyanin, E. K. (1995). *Progressive of Theoretical Physics Suppl* **118**, 35.
- Sklyanin, E. K. (1984). *Differential Geometry, Lie groups and Mechanics. VI Zap. Nauchn. Sem. Leningrad. (LOMI)* **133**, 236.